Self-consistent equation for an interacting Bose gas

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We consider interacting Bose gas in thermal equilibrium assuming a positive and bounded pair potential $V(r)$ such that $0 < \int d\mathbf{r}\, V(r) = a < \infty$. Expressing the partition function by the Feynman-Kac functional integral yields a classicallike polymer representation of the quantum gas. With the Mayer graph summation techniques, we demonstrate the existence of a self-consistent relation $\rho(\mu) = F(\mu - a\rho(\mu))$ between the density ρ and the chemical potential μ, valid in the range of convergence of Mayer series. The function F is equal to the sum of all rooted multiply connected graphs. Using Kac's scaling $V_\gamma(\mathbf{r}) = \gamma^3 V(\gamma r)$, we prove that in the mean-field limit $\gamma \to 0$, only the tree diagrams contribute and function F reduces to the free gas density. We also investigate how to extend the validity of the self-consistent relation beyond the convergence radius of the Mayer series (vicinity of Bose-Einstein condensation), and study the dominant corrections to the mean field. At the lowest order, the form of function F is shown to depend on a single polymer partition function for which we derive the lower and the upper bounds and on the resummation of ring diagrams which can be analytically performed.

I. INTRODUCTION

The interest for a better understanding of Bose-Einstein condensation has been strongly stimulated in recent years by the interesting experimental observations of condensates of cold atoms in traps [1,2].

Concerning rigorous results on the existence of Bose-Einstein condensation in an interacting gas with pair interactions, we quote the work of Lieb and Seiringer [3]. The authors show the existence of an off-diagonal long-range order in the ground state of a system of Bose particles, confined by an external potential in the dilute limit in which the Gross-Pitaewski equation becomes exact. Süto shows condensation for the trapped gas at nonzero temperature [4]. In Ref. [5], Lauwers, Verbeure, and Zagrebnov prove the existence of a Bose-Einstein phase transition for the homogeneous gas under the assumption that there is an energy gap at the bottom of the one-particle spectrum. However, to our knowledge, there is still no proof of Bose-Einstein condensation in the interacting gas when there is no trap and no gap. In the present work, we revisit this venerable many-body problem with a new point of view, i.e., the technique of quantum Mayer graphs.

In order to provide orientation and motivation for our approach, we recall some facts pertaining to the mean-field[1] Bose gas at a heuristic level. The Hamiltonian of N Bose

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[1] Here we use "mean field" in the sense of van der Waals, that is, collective effects of a long-range interaction. The same word is also used in the context of the dilute limit [3] leading to the Gross-Pitaewski regime. In fact, in the latter case, the situation is the opposite: dominant effects of the interaction are due to rarefied local binary collisions.

particles in a volume Λ, interacting with a constant repulsive potential of strength $a/|\Lambda|$, $a > 0$, reads

$$H_{\mathrm{mf},N} = H_{0,N} + \frac{a}{|\Lambda|}\frac{N(N-1)}{2}, \qquad (1)$$

where $H_{0,N}$ is the total kinetic energy. It leads to the free energy density

$$f_{\mathrm{mf}}(\beta,\rho) = f_0(\beta,\rho) + \frac{a}{2}\rho^2, \qquad (2)$$

where ρ is the particle density, β is related to the temperature T by $\beta = 1/k_B T$, k_B denotes the Boltzmann constant, and $f_0(\beta,\rho)$ is the free energy of the noninteracting Bose gas. Differenciating with respect to ρ yields the relation

$$\mu_0(\beta,\rho) = \mu_{\mathrm{mf}}(\beta,\rho) - a\rho \qquad (3)$$

between the chemical potential $\mu_0(\beta,\rho)$ of the free gas and the chemical potential $\mu_{\mathrm{mf}}(\beta,\rho)$ of the mean-field gas as functions of the density. Since the grand-canonical densities ρ_0 of the free gas and the density ρ_{mf} of the mean-field gas, considered as functions of the respective chemical potentials, are the inverse functions of $\mu_0(\beta,\rho)$ and $\mu_{\mathrm{mf}}(\beta,\rho)$ at fixed β, Eq. (3) is equivalent to the self-consistent equation

$$\rho_{\mathrm{mf}}(\beta,\mu) = \rho_0(\beta,\mu - a\rho_{\mathrm{mf}}(\beta,\mu)). \qquad (4)$$

In Eq. (4)

$$\rho_0(\beta,\mu) = \frac{1}{(2\pi\lambda^2)^{3/2}}\sum_{q=1}^{\infty}\frac{e^{\beta\mu q}}{q^{3/2}}, \qquad (5)$$

which is the well-known formula for the grand-canonical density of the free gas with

$$\lambda = \hbar \sqrt{\beta/m} \tag{6}$$

representing the thermal wavelength [6].

The series (5) converges for $\mu \leq 0$, so that the self-consistent equation (4) is meaningful whenever $\nu \equiv \mu - a\rho_{mf}(\beta,\mu) \leq 0$, namely for $\mu \leq \mu_c$ where the critical chemical potential μ_c is given by

$$\mu_c - a\rho_{mf}(\beta,\mu_c) = 0.$$

Thus μ_c has the value $\mu_c = a\rho_{0,c}(\beta)$, where $\rho_{0,c}(\beta) = \rho_0(\beta,\mu=0)$ is the critical density of the free gas.

At this point it is worth noting that the solution $\rho_{mf}(\mu)$ of Eq. (4) for $\mu \leq \mu_c$ can be extended to the range $\mu > \mu_c$ by continuity. Indeed, we differentiate Eq. (4) with respect to the chemical potential to obtain[2]

$$(\rho_{mf})'(\mu) = \frac{(\rho_0)'(\nu)}{1 + a(\rho_0)'(\nu)}. \tag{7}$$

One sees from Eq. (5) that $(\rho_0)'(\nu) = \infty$ for $\nu > 0$. Hence $(\rho_{mf})'(\mu) = 1/a$, $\nu > 0$, and requiring the continuity of the density at $\mu = \mu_c$ gives

$$\rho_{mf}(\mu) = \frac{\mu}{a}, \quad \mu > \mu_c. \tag{8}$$

Equations (4) and (8) define the density for all values of the chemical potential. For $\mu > \mu_c$, there is a Bose condensate of density $\rho_{mf}(\mu) - \rho_{0,c}$. These facts have been established with full mathematical rigor in several works [7–9], see Ref. [10] for a review. In particular, they are obtained for the Kac interparticle potential

$$V_\gamma(\mathbf{r}) = \gamma^3 V(\gamma \mathbf{r}) \tag{9}$$

in the scaling limit of γ tending to zero. When $\gamma \to 0$, the potential $V_\gamma(\mathbf{r})$ extends its range to infinity, whereas its amplitude tends to zero in such a way that the mean potential energy

$$\int d\mathbf{r} \, \gamma^3 V(\gamma \mathbf{r}) = a \tag{10}$$

stays constant. It is well known that in classical statistical mechanics this is the appropriate limit to rigorously recover the van der Waals mean-field theory [11]. Also the methods of Mayer graphs were used to calculate the corrections to the mean-field limit for small γ [12].

In this paper, we propose a similar approach to study an interacting Bose gas with nonsingular repulsive interactions. In Sec. II, we recall the "polymer" representation of the Bose gas in thermal equilibrium. Combining the Feynman-Kac functional integral representation of the Gibbs weight together with the decomposition of permutations into cycles, one finds the grand-canonical partition function in the clas-

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[2]We keep the temperature fixed and omit from now on β in the notation.

sicallike form of a gas of interacting polymers. Polymers are Brownian closed loops associated with a number of Bose particles belonging to a permutation cycle. Each loop has a self-energy and there is a loop-loop pair potential. In the space of polymers, all the techniques of classical statistical mechanics are available, in particular the analysis of the partition function and of the density with the Mayer diagrammatic techniques. In this way, we show in Sec. III how the mean-field equation (4) can be recovered by summing the tree graphs.

In Sec. IV, we treat an interacting gas with a general short-range repulsive two-body potential and establish that its density obeys an exact equation of the form

$$\rho(\mu) = F(\mu - a\rho(\mu)). \tag{11}$$

The function $F(\mu)$ is defined as the sum of multiply connected Mayer graphs. We prove with the help of the Penrose tree-graph inequality that the corresponding diagrammatic expansion of F is convergent at low density, namely if the chemical potential is sufficently negative. Equation (11) provides a generalization of the mean-field equation (4) to the interacting gas, i.e., if one introduces a Kac potential $V_\gamma(\mathbf{r})$, it is seen that $F_\gamma(\mu)$ reduces to the density $\rho_0(\mu)$ of the free gas as $\gamma \to 0$, thus recovering Eq. (4). The main aim then is to extend the validity of Eq. (11) to higher densities (hopefully up to a critical density), that is, to approach the Bose transition point from the dilute phase, as in the mean-field theory.

In Sec. V, we discuss the mathematical problems that arise at this point. The study of the critical point will require the control of the asymptotic behavior of the partition function of a single long repulsive polymer as well as of the mutual interactions between different polymers. One should note, however, that the polymers occurring in the representation of the Bose gas differ from the standard self-repelling classical polymers because of the specifically quantum mechanical "equal time interaction" introduced by the Feynman-Kac formula. Hence, the results of the theory of classical polymers cannot be used without further consideration. Investigating in this direction, we give the lower and the upper bounds on the partition function of a single polymer, indicating that Eq. (11) continues to hold for a range of chemical potentials larger than that assuring the convergence of the Mayer series. Moreover, we can calculate the effects of mutually interacting polymers at the lowest order in the Kac parameter γ by summing up the ring diagrams in a closed form. Concluding remarks are presented in Sec. VI. Proofs of some lemmas are relegated to Appendixes A and B, and Appendix C is devoted to the extension of our methods to an inhomogeneous Bose gas confined by an external potential.

II. THE POLYMER REPRESENTATION OF THE BOSE GAS

We consider bosons of mass m in three dimensions with the Hamiltonian

$$H_N = -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \Delta_i + \sum_{0 \leq i < j \leq N} V(\mathbf{r}_i - \mathbf{r}_j) \qquad (12)$$

enclosed in a box Λ, Δ being the Laplacian with Dirichlet conditions at the boundary of Λ. The pair potential $V(\mathbf{r})$ is short range, repulsive, and without any singularities:

$$V(\mathbf{r}) \geq 0, \quad \int d\mathbf{r}\, V(\mathbf{r}) \equiv a < \infty, \quad \sup_{\mathbf{r}} V(\mathbf{r}) \equiv \bar{V} < \infty. \qquad (13)$$

We assume, moreover, that its Fourier transform $\tilde{V}(\mathbf{k})$ is positive.

The polymer representation of the grand-partition function at chemical potential μ and inverse temperature β arises when the Gibbs statistical weight is expressed in terms of the Feynman-Kac path integral where quantum fluctuations are represented by the Brownian trajectories. The open trajectories associated with the exchange contributions are reorganized in closed loops (or polymers) containing several particles according to the decomposition of permutations into cycles. The result is that the grand-partition function of the Bose gas can be written in a classicallike form as (the so-called magic formula)

$$\Xi_\Lambda = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{n} d\mathcal{L}_i\, z(\mathcal{L}_i) \exp[-\beta U(\mathcal{L}_1, \ldots, \mathcal{L}_n)], \qquad (14)$$

provided that the suitable definitions of the phase space integration and of the interaction are given. In one form or another, this representation has been known since a long time in various contexts starting with the work of Ginibre on the convergence of quantum virial expansions [13]. It is also used to implement numerical simulations of the Bose gas [14]. The present form (14) has been derived and applied by Cornu [15] to the Coulomb systems, and we follow here the definitions given in Chap. V of Ref. [16]. A self-contained derivation can also be found in Ref. [17].

An element $\mathcal{L}$ of the phase space, called a loop or a polymer,

$$\mathcal{L} = (\mathbf{R}, q, \mathbf{X}(s), 0 \leq s \leq q) \qquad (15)$$

is specified by its position $\mathbf{R}$ in space, the number q of particles belonging to it, and its shape $\mathbf{X}(s)$. The q particles are located at the positions

$$\mathbf{r}_k = \mathbf{R} + \lambda \mathbf{X}(k-1), \quad k = 1, \ldots, q, \quad q+1 \equiv 1, \qquad (16)$$

and

$$\mathbf{r}_{k,k+1}(s) = \mathbf{R} + \lambda \mathbf{X}(k-1+s), \quad 0 \leq s \leq 1 \qquad (17)$$

is an open path joining the k particle at $\mathbf{r}_k$ to the $k+1$ particle at $\mathbf{r}_{k+1}$, where λ is the thermal wavelength (6). The loop can be viewed as an extended object at $\mathbf{R}$ which has internal degrees of freedom $(q, \mathbf{X})$ with q the number of particles belonging to a permutation cycle and $\mathbf{X}$ the shape of the loop, see Fig. 1.
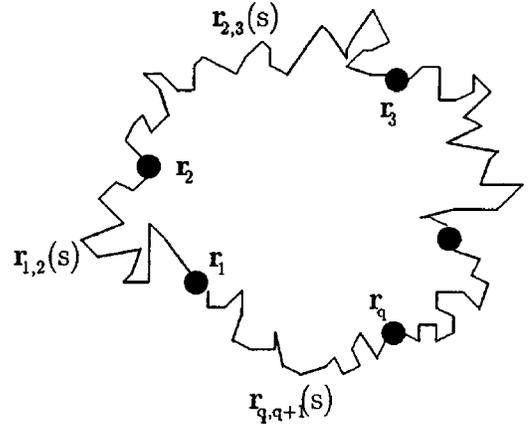
FIG. 1. A q particle loop.

The thermal wavelength λ (the only place where the Planck constant occurs) gives the extent of quantum fluctuations. The shape of the loop $\mathbf{X}$ is a Brownian bridge (a closed Brownian path), parametrized by the "time" s running in the "time interval" $[0,q]$ with $\mathbf{X}(0) = \mathbf{X}(q) = 0$. It is distributed according to the normalized Gaussian measure $\int D_q(\mathbf{X}) \ldots$, with covariance

$$\int D_q(\mathbf{X}) X_\mu(s_1) X_\nu(s_2) = \delta_{\mu\nu} q \left[\min\left(\frac{s_1}{q}, \frac{s_2}{q}\right) - \frac{s_1}{q}\frac{s_2}{q} \right], \qquad (18)$$

where X_μ, $\mu = 1,2,3$ are the Cartesian coordinates of $\mathbf{X}$. Integration on the phase space means integration over space and summation over all internal degrees of freedom of the loop,

$$\int d\mathcal{L} \cdots = \sum_{q=1}^{\infty} \int D_q(\mathbf{X}) \int_\Lambda d\mathbf{R} \cdots. \qquad (19)$$

Because of the Dirichlet boundary condition, the paths $\mathbf{X}$ are constrained to stay in the volume Λ, but we do not write this constraint explicitly since it will be removed later in the infinite volume limit.

The interaction energy of the two loops $\mathcal{L}_i, \mathcal{L}_j$ is the sum of pair Feynman-Kac potentials between the particles associated to the loops,

$$V(\mathcal{L}_i, \mathcal{L}_j) = \sum_{k=1}^{q_i} \sum_{\ell=1}^{q_j} \int_0^1 ds\, V(\mathbf{R}_i + \lambda \mathbf{X}_i(k-1+s)$$

$$-\mathbf{R}_j - \lambda \mathbf{X}_j(\ell-1+s))$$

$$= \int_0^{q_i} ds_i \int_0^{q_j} ds_j\, \tilde{\delta}(s_i - s_j) V(\mathbf{R}_i + \lambda \mathbf{X}_i(s_i)$$

$$-\mathbf{R}_j - \lambda \mathbf{X}_j(s_j)). \qquad (20)$$

In Eq. (20), the distribution

$$\tilde{\delta}(s) = \sum_{n=-\infty}^{\infty} e^{2i\pi ns} \qquad (21)$$

is the periodic Dirac function of period 1.

The activity $z(\mathcal{L})$ of a loop is related to the chemical potential μ of the particles by

$$z(\mathcal{L}) = \frac{z^q}{q(2\pi q\lambda^2)^{3/2}} \exp[-\beta U(\mathcal{L})], \quad z = e^{\beta\mu}. \quad (22)$$

It incorporates the interactions $U(\mathcal{L})$ of the particles in the same loop (the self-energy of the loop),

$$U(\mathcal{L}) = \frac{1}{2} \int_0^q ds_1 \int_0^q ds_2 \, \tilde{\delta}(s_1 - s_2) V(\lambda[\mathbf{X}(s_1) - \mathbf{X}(s_2)])$$

$$- \frac{1}{2} q V(0). \quad (23)$$

The last term substracts out the self-energy of the particles. $U(\mathcal{L})$ can as well be written as

$$U(\mathcal{L}) = \frac{1}{2} \int_0^q ds_1 \int_0^q ds_2 \, \tilde{\delta}(s_1 - s_2)(1 - \delta^{Kr}_{[s_1],[s_2]})$$

$$\times V(\lambda[\mathbf{X}(s_1) - \mathbf{X}(s_2)]) \geq 0, \quad (24)$$

making manifest that $U(\mathcal{L})$ is a positive quantity (in the Kronecker δ symbol $[s]$ denotes the integer part of s). As a direct consequence, one gets the bound

$$0 \leq z(\mathcal{L}) \leq \frac{e^{\beta\mu q}}{q(2\pi q\lambda^2)^{3/2}} \equiv z^{(0)}(q). \quad (25)$$

From the structure (14) of the partition function and the above definitions, it is clear that the calculation rules of the classical statistical mechanics apply to the system of loops. We shall take advantage of this fact to analyze the system of loops first and derive from there the results for the original quantum gas of particles. In particular, all the powerful techniques of Mayer graphs are available to expand the loop density and the loop correlations in powers of the loop activities $z(\mathcal{L})$. It is convenient to introduce the abbreviated notation $\mathcal{L}_i = i$ and $d\mathcal{L}_i = di$. Mayer bonds are defined by

$$f(i,j) = e^{-\beta V(i,j)} - 1 \quad (26)$$

and weights at vertices by $z(i)$ (22). Integration at vertices di has to be performed according to Eq. (19).

Notice that the bond is integrable over space since from the positivity of $V(\mathbf{r})$,

$$|f(i,j)| \leq \beta V(i,j) \quad (27)$$

and from Eq. (20),

$$\int d\mathbf{R}_j |f(\mathcal{L}_i, \mathcal{L}_j)| \leq \beta \int d\mathbf{R}_j \, V(\mathcal{L}_i, \mathcal{L}_j)$$

$$= \beta \int_0^{q_i} ds_i \int_0^{q_j} ds_j \, \tilde{\delta}(s_i - s_j) \int d\mathbf{R} \, V(\mathbf{R})$$

$$= \beta a q_i q_j. \quad (28)$$

The loop density $\rho_{\text{loop}}(\mathcal{L})$ is given by the standard expansion

$$\rho_{\text{loop}}(1) = \sum_{n=1}^{\infty} n B_n(1), \quad (29)$$

where

$$B_n(1) = \frac{1}{n!} \int d2 \cdots dn z(1) z(2) \cdots z(n) u(1,2,\ldots,n), \quad (30)$$

and

$$u(1,2,\ldots,n) = \sum_{\Gamma_n} \prod_{(i,j) \in \Gamma_n} f(i,j) \quad (31)$$

is the Ursell function. The sum runs over all labeled connected graphs Γ_n with n vertices. We have directly written the loop density (29) in the infinite volume limit, namely extending the spatial integrals $d\mathbf{R}_2, \ldots, d\mathbf{R}_n$ at vertices $2, \ldots, n$ over the whole space, whereas the vertex 1 (the root point of the graph) carries no spatial integration. The existence of the infinite volume limit of the individual Mayer graphs follows from the translation invariance and integrability of the Mayer bond as in the classical case. As a consequence, $\rho_{\text{loop}}(\mathcal{L}) = \rho_{\text{loop}}(q, \mathbf{X})$ does not depend on the location $\mathbf{R}$ of the loop $\mathcal{L}$ in space.

Finally, to obtain the original particle density $\rho(\mu)$ from the loop density, we have to sum $\rho_{\text{loop}}(\mathcal{L})$ over the internal variables of $\mathcal{L}$,

$$\rho(\mu) = \sum_{q=1}^{\infty} q \int D_q(\mathbf{X}) \rho_{\text{loop}}(q, \mathbf{X}), \quad (32)$$

the additional q factor takes into account that the loop $\mathcal{L}$ carries q particles.

Although classical methods have been used, it is important to stress the difference between the loop representation of the equilibrium state of the quantum Bose gas and that of a gas of genuine classical polymers. First, the chemical potential μ is not the variable which is thermodynamically conjugate to the polymer number, but to the original particle number. Moreover, the loop interactions differ from the classical polymer interactions by the quantum mechanical "equal time prescription," which originates from the Feynman-Kac formula. This equal time prescription is manifested in Eqs. (20) and (23) by the occurrence of the periodic delta function $\tilde{\delta}(s_1 - s_2)$. In the classical interaction, every segment of the polymer interacts pairwise with any other segment, which would correspond to the potentials (20) and (23) without the equal time prescription. The purely classical gas of point particles, interacting by means of the two-body potential $V(\mathbf{r})$, is recovered if the thermal length λ is set equal to zero in (20) and if only the terms with $q=1$ (Boltzmann statistics) are retained.
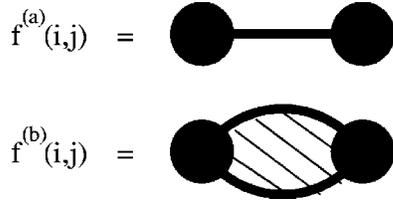
$$f^{(a)}(i,j) \;=$$

$$f^{(b)}(i,j) \;=$$

FIG. 2. The bonds $f^{(a)}(i,j)$ and $f^{(b)}(i,j)$.

III. MEAN-FIELD LIMIT AND TREE-GRAPH SUMMATION

Let us first show how the self-consistent mean-field Bose gas is recovered in the diagrammatic analysis. To this end, we split the bond $f(i,j)$ into its part linearized in the potential and higher-order terms,

$$f(i,j)=f^{(a)}(i,j)+f^{(b)}(i,j),$$

$$f^{(a)}(i,j)=-\beta V(i,j),$$

$$f^{(b)}(i,j)=e^{-\beta V(i,j)}-1+\beta V(i,j). \tag{33}$$

These bonds are represented in Fig. 2.

This simply enlarges the previous class of graphs (Γ) to graphs (still denoted by Γ) where each bond can be either $f^{(a)}(i,j)$ or $f^{(b)}(i,j)$.

In the mean-field limit $\gamma \to 0$, the contributions of order $O(1)$ in γ due to the scaled potential $\gamma^3 V(\gamma \mathbf{r})$ will come from the linearized bonds

$$f^{(a)}_\gamma(i,j)=-\beta\gamma^3 \int_0^{q_i} ds_i \int_0^{q_j} ds_j \, \tilde\delta(s_i-s_j)$$

$$\times V(\gamma(\mathbf{R}_i-\mathbf{R}_j)+\gamma\lambda[\mathbf{X}_i(s_i)-\mathbf{X}_j(s_j)]) \tag{34}$$

and from the vertex weights

$$z^{(0)}(i)=\frac{z^{q_i}}{q_i(2\pi q_i\lambda^2)^{3/2}}. \tag{35}$$

The bond $f^{(a)}_\gamma(i,j)$ will give contributions of order $O(1)$ since under scaling (10) its total spatial integral

$$\int d\mathbf{R}_j f^{(a)}_\gamma(i,j)=-\beta a q_i q_j \tag{36}$$

is independent of γ. In the activity (22), we simply disregard the self-energy $U_\gamma(\mathcal{L})$ since the latter is $O(\gamma^3)$.

Proposition 1. The density $\rho_{\text{tree}}(\mu)$ calculated as the sum of all the tree graphs with bonds (34) and vertices (35) verifies the mean-field equation (4).

Proof. Consider a rooted tree-graph T_{n+1} with vertices $(0,1,\ldots,n)$ for which the root point is of degree 1 (0 is the label of the root point),[3] see Fig. 3. Hence the root point is linked to the rest of the graph by a single bond, say $f^{(a)}(0,1)$. Call T_n the subgraph of T_{n+1} with vertices $(1,2,\ldots,n)$ and

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[3]The degree of a point is the number of lines incident at this point.
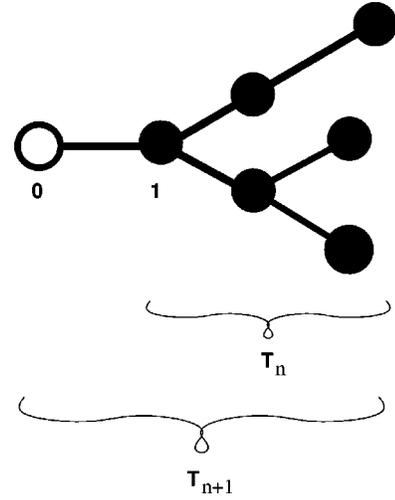
FIG. 3. Rooted tree T_{n+1} with subtree T_n.

$t_n(1)$ the value of this subgraph once integrated on the vertices $2,\ldots,n$. Then the value $t_{n+1}(\mathcal{L}_0)$ of the rooted graph T_{n+1} is

$$t_{n+1}(\mathcal{L}_0)=-\beta z^{(0)}(q_0)\int d\mathcal{L}_1\, V_\gamma(\mathcal{L}_0,\mathcal{L}_1)t_n(\mathcal{L}_1). \tag{37}$$

The vertex of the graph T_n linked to the root point 0 can have n different labels. The resulting tree graphs are different because they contain different links with the root point. Moreover, because of translational invariance, $t_n(1)=t_n(q_1,\mathbf{X}_1)$ does not depend on the position $\mathbf{R}_1$ of the loop $\mathcal{L}_1$. Thus the spatial integration over $\mathbf{R}_1$ can be performed on $V_\gamma(\mathcal{L}_0,\mathcal{L}_1)$ as in Eq. (36) so that the total contribution reads

$$nt_{n+1}(\mathcal{L}_0)=z^{(0)}(q_0)(-\beta a q_0)$$

$$\times\left[\sum_{q_1=1}^\infty q_1\int D(\mathbf{X}_1)nt_n(q_1,\mathbf{X}_1)\right]. \tag{38}$$

According to Eqs. (29) and (32), the quantity in the bracket is precisely the contribution to the particle density $\rho_{\text{tree}}(\mu)$ of the graph T_n. Therefore the sum of all the tree graphs having a root point of degree 1 is $z^{(0)}(q_0)[-\beta a q_0 \rho_{\text{tree}}(\mu)]$. The sum of all the tree graphs with root point of degree n is $z^{(0)}(q_0)[-\beta a q_0 \rho_{\text{tree}}(\mu)]^n/n!$ (the factor $1/n!$ takes care of the fact that the labeling of the vertices belonging to different branches attached to the root point can be permuted without giving rise to new Mayer graphs). Finally, summing on all the trees rooted at the point 0 gives the density of loops. According to Eq. (32), to obtain the particle density we still have to sum on the internal variables $q_0,\mathbf{X}_0$ of the root loop with a factor q_0. Hence using Eq. (35), we find

$$\rho_{\text{tree}}(\mu)=\sum_{q_0=1}^\infty \sum_{n=0}^\infty \frac{q_0 z^{(0)}(q_0)}{n!}(-\beta a q_0 \rho_{\text{tree}}(\mu))^n$$

$$=\sum_{q_0=1}^\infty \frac{\exp(\beta[\mu-a\rho_{\text{tree}}(\mu)])}{(2\pi q_0\lambda^2)^{3/2}}=\rho_0(\mu-a\rho_{\text{tree}}(\mu)),$$

$$\tag{39}$$

which is the mean-field equation (4).

We now come back to the general Mayer series and prove its convergence at low density.

Proposition 2. The Mayer series (29), (32) converges for $\mu \leq -a\rho_{0,c}$ with $\rho_{0,c} = \rho_0(\beta, \mu = 0)$ the critical density of the free gas.

Proof. We use the Penrose tree-graph inequality for positive potentials which states that the sum of the tree graphs provides an upper bound for the Ursell function [18]

$$u(1,2,\ldots,n) \leq \sum_{T_n} \prod_{(i,j) \in T_n} |f(i,j)| \leq \sum_{T_n} \prod_{(i,j) \in T_n} (\beta V(i,j)),$$
(40)

the second inequality being a consequence of Eq. (27). Moreover, if in the series (29) we use the inequality (25), we see that the tree summation with bonds $|f^{(a)}(i,j)| = \beta V(i,j)$ and vertices $z^{(0)}(i)$ provides also an upper bound. According to Proposition 1, the latter series sums up to a function $\bar{\rho}(\mu)$ that obeys

$$\bar{\rho}(\mu) = \rho_0(\mu + a\bar{\rho}(\mu)).$$
(41)

This is the mean-field equation of a Bose system for the negative potential $-V(\mathbf{r})$. It has a finite solution provided $\mu + a\bar{\rho}(\mu) \leq 0$ which is equivalent to $\mu \leq -a\rho_{0,c}$. We conclude that

$$\rho(\mu) \leq \bar{\rho}(\mu) < \infty, \quad \mu \leq -a\rho_{0,c}.$$
(42)

Notice that in the case of the scaled potential $V_\gamma(\mathbf{r})$, the convergence of the series defining $\rho_\gamma(\mu)$ is not only absolute but also uniform with respect to γ. This follows from the fact that in the evaluation of the tree-graph contributions one encounters only the integrated bonds,

$$\int d\mathbf{R}_j |f_\gamma^{(a)}(i,j)| = \beta a q_i q_j.$$

The Penrose inequality yields thus a γ-independent upper bound. Now we show that in Kac limit, the density converges to the mean-field value.

Proposition 3. Let $\rho_\gamma(\mu)$ be the density associated with the scaled potential $V_\gamma(\mathbf{r}) = \gamma^3 V(\gamma \mathbf{r})$. Then

$$\lim_{\gamma \to 0} \rho_\gamma(\mu) = \rho_{\mathrm{mf}}(\mu), \quad \mu \leq -a\rho_{0,c}.$$
(43)

Proof. Consider first a graph Γ_n with vertices $1,2,\ldots,n$ (rooted at point 1) which is not a tree, i.e., it contains at least one cycle. Delete some bonds in such a way that the graph thus obtained is a connected tree T_n with value $t_n(1)$. In order to find an upper bound for the value $g_{n,\gamma}(1)$ of Γ_n, we use the inequalites

$$|f_\gamma(i,j)| \leq \beta V_\gamma(i,j),$$
(44)

for the bonds remaining in the tree, and

$$|f_\gamma(i,j)| \leq \gamma^3 \beta q_i q_j \bar{V},$$
(45)

for each deleted bond. Notice that Eq. (45) is obtained by replacing the pair potential in Eq. (34) by its supremum $\bar{V}$. Moreover, the activity $z(i)$ is bounded by $z^{(0)}(i)$ (25). The spatial integration of a bond (i,j) of the tree yields the factor $\beta a q_i q_j$ as before so that with Eq. (45) a vertex (i) of degree k_i in Γ_n receives a factor $q_i^{k_i}$ multiplied by the activity $z^{(0)}(q_i)$. If ℓ bonds have been deleted, the resulting inequality has the form

$$|g_{n,\gamma}(1)| \leq (\gamma^3 \bar{V})^\ell (\beta a)^{n-1}$$

$$\times \sum_{q_2,\ldots,q_n=1}^{\infty} q_1^{k_1} \cdots q_n^{k_n} z^{(0)}(q_1) \cdots z^{(0)}(q_n).$$
(46)

In view of the exponential factor $e^{\beta \mu q}$ in Eq. (25), the q series in Eq. (46) is convergent for $\mu < 0$. Hence the presence of one cycle in Γ_n implies that $g_{n,\gamma}(1)$ tends to zero not slower than γ^3 for $\gamma \to 0$ (and as $\gamma^{3\ell}$ if Γ_n has ℓ cycles).

If Γ_n is a tree, we decompose the Mayer bonds as in Eq. (33). Since

$$|f^{(b)}(i,j)| \leq \frac{1}{2}(\beta V(i,j))^2 \leq \frac{1}{2}\gamma^3 \beta q_i q_j \bar{V}(\beta V(i,j)),$$
(47)

all the contributions to Γ_n containing $f^{(b)}(i,j)$ bonds vanish as $\gamma \to 0$.

As we have seen from the Penrose estimate and from Proposition 2, the Mayer series constituting $\rho_\gamma(\mu)$ is absolutely uniformly convergent with respect to γ. By dominated convergence, the limit of the diagrammatic sum can be calculated term by term. Therefore we are left with the sum of trees where all the bonds are of the type $f_\gamma^{(a)}(i,j)$ and where the activities $z_\gamma(i)$ can be replaced by $z^{(0)}(i)$ as $\gamma \to 0$. This is exactly the situation as in Proposition 1 thus proving Proposition 3.

IV. THE SELF-CONSISTENT EQUATION FOR THE INTERACTING GAS

The reasoning which led to Eq. (39) was restricted to the tree diagrams. We now show how it can be generalized to provide an implicit equation for the exact density $\rho(\mu)$. To this end, we consider the complete set of expanded Mayer graphs with the two types of bonds $f^{(a)}(i,j)$ and $f^{(b)}(i,j)$, defined in Eq. (33), and call $f^{(a)}(i,j)$ a single interaction bond. A graph is said to be multiply connected[4] if it cannot be disconnected by cutting a single interaction bond. Then we define $I(\mathcal{L})$ to be the value of the sum of all multiply connected graphs with one root point $\mathcal{L}$, see Fig. 4.

Because of translation invariance, $I(\mathcal{L})$ does not depend on the position $\mathbf{R}$ of $\mathcal{L}$. Denoting here the chemical potential by ν, we define the function

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[4]In the context of Feynman diagrams, such graphs are also called one-line irreducibles.
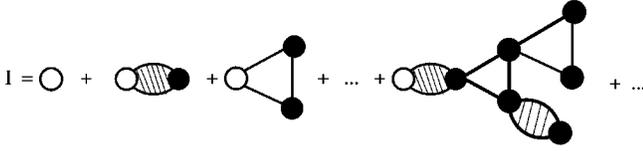
FIG. 4. The sum of multiply connected graphs.

$$F(\nu) = \sum_{q=1}^{\infty} q \int D_q(\mathbf{X}) I(\mathcal{L}). \quad (48)$$

For $\nu \leqslant -a\rho_{0,c}$, the function $F(\nu)$ is represented by a convergent sum since it is a subseries of the absolutely convergent Mayer expansion (Proposition 2).

Proposition 4. For $\mu \leqslant -a\rho_{0,c}$, $\rho(\mu)$ verifies the equation

$$\rho(\mu) = F(\mu - a\rho(\mu)). \quad (49)$$

Proof. To establish Eqs. (48) and (49), we proceed as in Proposition 1. Considering now a general structure of Mayer graphs, we say that a vertex (i) in a graph has a star structure if there is a number of incident single interaction lines at vertex (i) such that cutting any one of them disconnects the graph. Consider first a vertex (i) having a star structure consisting of one single interaction line. This interaction line, say $f^{(a)}(i,1)$, links the vertex (i) to a subgraph Γ_n with vertices $(1,2,\ldots,n)$; Γ_n has no other links with the rest of the graph, see Fig. 5. Once integrated on the points $(2,\ldots,n)$, its value $g_n(1)$ does not depend on the position $\mathbf{R}_1$ of the loop $\mathcal{L}_1$. Thus the spatial integration over $\mathbf{R}_1$ can be performed on the bond $[-\beta V(\mathcal{L}_i, \mathcal{L}_1)]$ yielding the factor $(-\beta a q_i q_1)$. As in (38), its total contribution to the vertex (i) equals

$$z(\mathcal{L}_i)(-\beta a q_i) \left[\sum_{q_1=1}^{\infty} q_1 \int D_{q_1}(\mathbf{X}_1) n g_n(q_1, \mathbf{X}_1) \right]. \quad (50)$$

But from Eq. (32), the quantity in the square bracket is the contribution of the graph Γ_n to the exact density. Thus the sum of all such graphs contributes to the vertex (i) as $z(\mathcal{L}_i)(-\beta a q_i \rho(\mu))$. Continuing the reasoning along the lines of the proof of Proposition 1, we consider next the sum of all star structures at (i) having n single interaction lines. Their contribution equals $z(\mathcal{L}_i)[-\beta a q_i \rho(\mu)]^n/n!$. Finally,
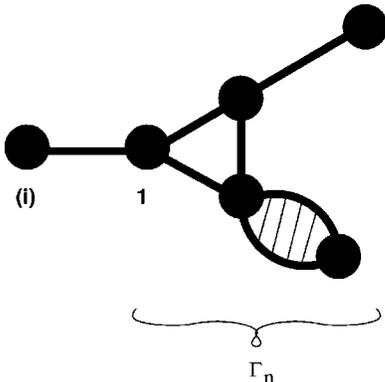
FIG. 5. Star structure at vertex (i) and subgraph Γ_n.

summing all possible star structures at (i), we arrive at the formula for an effective activity

$$z^*(\mathcal{L}_i) = z(\mathcal{L}_i) e^{-\beta a \rho(\mu) q_i} = \frac{e^{\beta(\mu - a\rho(\mu))q_i}}{q_i(2\pi q_i \lambda^2)^{3/2}} e^{-\beta U(\mathcal{L}_i)}. \quad (51)$$

This is precisely the previous activity $z(\mathcal{L}_i)$ (22) evaluated at the shifted chemical potential $\nu = \mu - a\rho(\mu)$.

From this analysis, we see that $\rho(\mu)$ can as well be obtained by summing all multiply connected graphs with bonds $f^{(a)}(i,j)$ and $f^{(b)}(i,j)$ with effective activities $z^*(\mathcal{L}_i)$ at vertices. Indeed, by construction, any graph in these series is an original Mayer graph. Conversely, each Mayer graph does appear therein. This can be shown by the following reasoning. Consider any connected graph made of the bonds $f^{(a)}(i,j)$ and $f^{(b)}(i,j)$ and remove all single interaction lines $f^a(i,j)$ whose presence makes it not mutliply connected (by definition removal of such a line disconnects the root point from a part of the graph). Then the remaining subgraph connected to the root point is a multiply connected graph from which the original Mayer graph is obtained in a unique way by forming star structures. This concludes the proof since, in view of formula (51), by forming $F(\nu)$ according to these rules we find the density equal to the value of this function at the shifted argument $\nu = \mu - a\rho(\mu)$.

As a corollary of Proposition 3, we have that if $F_\gamma(\nu)$ is calculated with the scaled potential $V_\gamma(\mathbf{r})$, then $\lim_{\gamma \to 0} F_\gamma(\nu) = \rho_0(\nu)$.

We can write $I(\mathcal{L})$ appearing in the definition (48) of function $F_\gamma(\nu)$ as the sum

$$I(\mathcal{L}) = z(\mathcal{L})[1 + I_{\text{cycle}}(\mathcal{L})] \quad (52)$$

of the pure root point term plus all the multiply connected graphs containing cycles and $f^{(b)}(i,j)$ bonds. It is clear from the proof of Proposition 3 that the mean-field density arises solely from the root point $z(\mathcal{L})$, whereas $I_{\text{cycle}}(\mathcal{L})$ becomes vanishingly small in the mean-field limit.

All the results presented so far are valid within the convergence radius of the Mayer series determined by $\mu < -a\rho_{0,c}$. The question then arises as to whether the density, as solution of Eq. (49), can be extended to larger values of μ (as was possible in the strict mean-field case). An investigation of this question and of the mathematical difficulties involved is presented in the following section.

V. THE BOSE GAS BEYOND THE MEAN FIELD

A. Vertex contribution: The single polymer partition function

In this section, we study the structure of function $F(\nu)$ in a more detailed way. The first term $f^{(0)}(\nu)$ of the expansion of $F(\nu)$ in multiply connected graphs corresponds to the root point $z(\mathcal{L})$, which is a single vertex contribution,

$$f^{(0)}(\nu) = \sum_{q=1}^{\infty} q \int D_q(\mathbf{X}) z(\mathcal{L}) = \frac{1}{(2\pi\lambda^2)^{3/2}} \sum_{q=1}^{\infty} \frac{e^{\beta\nu q}}{q^{3/2}} \kappa(q).$$

(53)

Here we have introduced the mean value of the Boltzmann factor

$$\kappa(q) = \langle e^{-\beta U} \rangle_q \equiv \int D_q(\mathbf{X}) e^{-\beta U(q,\mathbf{X})}$$

(54)

of a single closed polymer carrying the repulsive energy (23). The Brownian bridge measure defining the average $\langle \cdots \rangle_q$ is normalized to 1, namely if $U(q,\mathbf{X}) = 0$, one has $\kappa(q) = 1$ and $f^{(0)}(\nu)$ reduces then to the free gas density (5). Moreover, for positive potentials we have $0 \le \kappa(q) \le 1$. Clearly, the radius of convergence of the series (53) is determined by the asymptotic behavior of $\kappa(q)$ for $q \to \infty$. The following bounds can be established.

Lemma 1. There exists ν_+ such that

$$\kappa(q) \ge e^{-\beta q \nu_+}.$$

(55)

Proof. This lower bound follows from Jensen inequality

$$\kappa(q) \ge \exp(-\beta \langle U \rangle_q),$$

(56)

where

$$\langle U \rangle_q = \int D_q(\mathbf{X}) U(q,\mathbf{X})$$

(57)

is the average energy of the polymer. The value of ν_+ results from a direct calculation of $\langle U \rangle_q$ presented in Appendix A.

There is also an upper bound.

Lemma 2. Let r be a fixed integer $r \ge 2$. There exists ν_- (depending on r) such that $0 < \nu_- < \nu_+$ and

$$\kappa(qr) \le r^{3/2} \exp(-\beta q r \nu_-).$$

(58)

Proof. The upper bound is derived by splitting the closed loop $\mathbf{X}(s)$, $0 \le s \le q$, $\mathbf{X}(0) = \mathbf{X}(q) = 0$, into the union of two open Brownian paths $\mathbf{X}_1(s)$, $0 \le s \le q_1$ and $\mathbf{X}_2(s)$, $q_1 \le s \le q$, and disregarding the (positive) interactions between the two paths $\mathbf{X}_1$ and $\mathbf{X}_2$. The details and the form of ν_- are given in Appendix B.

The lemmas imply that $f^{(0)}(\nu)$ is finite if $\nu \le \nu_-$ and diverges for $\nu > \nu_+$. Let us suppose for a moment that there exists a critical value ν_c such that

$$f^{(0)}(\nu) < \infty, \quad \nu \le \nu_c,$$

$$f^{(0)}(\nu) = \infty, \quad \nu > \nu_c.$$

(59)

In this case, there will be a critical density $\rho_c = f^{(0)}(\nu_c)$ and a critical chemical potential $\mu_c = a\rho_c + \nu_c$. From Lemmas 1 and 2, one has necessarily $\nu_- \le \nu_c < \nu_+$.

The determination of the possible existence of ν_c requires the knowledge of the exact asymptotic behavior of $\kappa(q)$ as $q \to \infty$. In this respect, let us make the following comment concerning the theory of classical polymers, defined as those interacting via the standard pairwise repulsion,
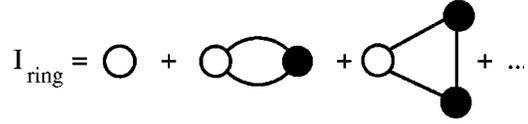
FIG. 6. The sum of rings.

$$U_{\mathrm{cl}}(\mathcal{L}) = \frac{1}{2} \int_0^q ds_1 \int_0^q ds_2 \, V(\mathbf{X}(s_1) - \mathbf{X}(s_2)).$$

(60)

It is firmly established, although not rigorously proved, that the normalized partition function of a single classical closed polymer behaves (in three dimensions) as [19]

$$\kappa_{\mathrm{cl}}(q) = \int D_q(\mathbf{X}) e^{-\beta U_{\mathrm{cl}}(\mathcal{L})} \sim C \frac{e^{-\beta A q}}{q^{3(\nu_{\mathrm{pol}} - 1/2)}}, \quad q \to \infty,$$

(61)

where A is a constant depending on the choice of the potential $V(\mathbf{r})$, and $\nu_{\mathrm{pol}} = 0{,}589$ is the universal critical exponent for a swollen polymer. Inserting this asymptotic behavior in the series (53) gives $\nu_c = A$ and a finite critical density ρ_c since the series is convergent at $\nu = \nu_c$. It is an open question to find out whether a similar situation holds for the "quantum" polymers subjected to the "equal time" interaction (23).

B. Bond contributions: Interacting polymers

The interaction between different polymers occurs in multiply connected graphs having the bonds $f^{(a)}(i,j)$ or $f^{(b)}(i,j)$. It turns out that the subseries $I_{\mathrm{ring}}(\mathcal{L})$ of $I_{\mathrm{cycle}}(\mathcal{L})$ defined as the sum of all multiply connected graphs having exactly one cycle of interaction bonds can be summed in a closed form, see Fig. 6 [the second term of the series corresponds to the quadratic term $(\beta V)^2(i,j)/2$ in the expansion of $f^{(b)}(i,j)$, (33)]. The result is

$$I_{\mathrm{ring}}(\mathcal{L}) = \frac{1}{2} \int d\mathbf{k} \int_0^q ds \int_0^q dt \, e^{i\lambda \mathbf{k} \cdot (\mathbf{X}(s) - \mathbf{X}(t))}$$

$$\times \sum_{n=-\infty}^{\infty} \frac{(\beta \tilde{V}(\mathbf{k}))^2 \alpha_n^2(\mathbf{k})}{1 + \beta \tilde{V}(\mathbf{k}) \alpha_n^2(\mathbf{k})} e^{2i\pi n(s-t)}.$$

(62)

Here $\tilde{V}(\mathbf{k})$ is the Fourier transform of the potential, and the positive coefficients $\alpha_n^2(\mathbf{k})$,

$$\alpha_n^2(\mathbf{k}) = \sum_{q=0}^{\infty} q \int_0^q ds \int D_q(\mathbf{X}) z(\mathcal{L}) e^{i\lambda \mathbf{k} \cdot \mathbf{X}(s)} e^{2i\pi ns}, \quad (63)$$

come from the summation of the internal degrees of freedom of loops at vertices. A derivation of formula (62) can be found in Ref. [20] where the effective loop-loop potential is calculated as the sum of all chain graphs. $I_{\mathrm{ring}}(\mathcal{L})$ is given precisely by formula (84) in Ref. [20], restricted to a single species of bosonic particles and with the Fourier transform $4\pi/|\mathbf{k}|^2$ of the Coulomb potential replaced by the present

short-range potential $\tilde{V}(\mathbf{k})$ [in Ref. [20] the coefficients $4\pi\beta\alpha_n^2(\mathbf{k})$ are noted $\kappa^2(\mathbf{k},n)$].

We now check that $I_{\mathrm{ring}}(\mathcal{L})$ is also well defined in the range $\nu<\nu_-$ [or possibly $\nu<\nu_c$ if there is a critical value ν_c (59)]. As the modulus of the phase factors in Eq. (62) equals 1, the positivity of $\beta\tilde{V}(\mathbf{k})\alpha_n^2(\mathbf{k})$ implies the inequality

$$I_{\mathrm{ring}}(\mathcal{L}) \leqslant \frac{q^2}{2}\int d\mathbf{k}(\beta\tilde{V}(\mathbf{k}))^2\sum_{n=-\infty}^{\infty}\alpha_n^2(\mathbf{k}). \qquad (64)$$

From Eqs. (63) and (21)

$$\sum_{n=-\infty}^{\infty}\alpha_n^2(\mathbf{k}) = \sum_{q=1}^{\infty} q\int D_q(\mathbf{X})z(\mathcal{L})\left[\int_0^q ds\, e^{i\lambda\mathbf{k}\cdot\mathbf{X}(s)}\tilde{\delta}(s)\right]$$

$$\leqslant f^{(1)}(\nu). \qquad (65)$$

We have defined

$$f^{(k)}(\nu) = \sum_{q=1}^{\infty} q^{k+1}\int D_q(\mathbf{X})z(\mathcal{L}) = \sum_{q=1}^{\infty} q^k\frac{e^{\beta\nu q}}{(2\pi q\lambda^2)^{3/2}}\kappa(q), \qquad (66)$$

and the inequality follows from the fact that the bracket $[\cdots]$ is less than or equal to q. Hence the contribution of the ring diagrams $F_{\mathrm{ring}}(\nu)$ to the function $F(\nu)$,

$$F_{\mathrm{ring}}(\nu) = \sum_{q=1}^{\infty} q\int D(\mathbf{X})z(\mathcal{L})I_{\mathrm{ring}}(\mathcal{L}), \qquad (67)$$

is bounded by

$$F_{\mathrm{ring}}(\nu) \leqslant \frac{1}{2}\int d\mathbf{k}(\beta\tilde{V}(\mathbf{k}))^2 f^{(1)}(\nu)f^{(2)}(\nu), \qquad (68)$$

where inequality (25) has been also used. In view of Lemma 1, it is clear that $f^{(k)}(\nu)$ (66) is finite for $\nu<\nu_-$, and thus $F_{\mathrm{ring}}(\nu)$ is also well defined in this extended range of chemical potentials.

The ring contribution is expected to be small for a scaled potential V_γ when $\gamma\to0$. Indeed if the Fourier transform $\tilde{V}_\gamma(\mathbf{k})=\tilde{V}(\mathbf{k}/\gamma)$ of V_γ is introduced in Eq. (68) and the $\mathbf{k}$ integration variable is changed into $\mathbf{k}=\gamma\mathbf{p}$, we find immediately

$$F_{\mathrm{ring},\gamma}(\nu) \leqslant \frac{\gamma^3}{2}\int d\mathbf{p}(\beta\tilde{V}(\mathbf{p}))^2 f_\gamma^{(1)}(\nu)f_\gamma^{(2)}(\nu) = O(\gamma^3),$$

$$\nu<\nu_-. \qquad (69)$$

In Eq. (69), $f_\gamma^{(k)}(\nu)$ are still dependent on γ through the single polymer partition function $\kappa_\gamma(q)$.

If one introduces the scaled potential in the full expression (62) (setting $\mathbf{k}=\gamma\mathbf{p}$), one observes the subtle dependence of the parameter γ. There is an overall γ^3 prefactor that manifests the smallness of the scaled potential as in the classical case. A γ factor occurs also in the phase in the integrands of Eqs. (62) and (63) in the combination $\gamma\lambda$,

representing the ratio of the thermal wavelength and the potential range. The importance of these phases will therefore depend on the value of this ratio. Finally, γ appears in the loop activity $z_\gamma(\mathcal{L})$ in Eq. (63). For γ small and $\gamma\lambda\ll1$, we can replace the above mentioned phase factors by 1 and approximate

$$F_{\mathrm{ring}}(\nu) \sim \frac{\gamma^3}{2}f_\gamma^{(2)}(\nu)\int d\mathbf{p}\frac{(\beta\tilde{V}(\mathbf{p}))^2 f_\gamma^{(1)}(\nu)}{1+\beta\tilde{V}(\mathbf{p})f_\gamma^{(1)}(\nu)}, \quad \gamma\to0. \qquad (70)$$

This will be the dominant term in the expansion of $I_{\mathrm{cycle}}(\nu)$ since the terms with $k>1$ cycles will obtain an overall γ^{3k} prefactor.

VI. CONCLUDING REMARKS

The use of quantum Mayer graphs proved useful in discovering the existence of an implicit equation (49), defining the density of an interacting Bose gas as function of its chemical potential. The knowledge of the precise form of the equation requires the resummation of all multiply connected diagrams including the single vertex contribution. It should be recalled that in a series of papers, Lee and Yang have developed a diagrammatic formalism for the quantum statistical mechanical, many-body problem that enables us to calculate the thermodynamical quantities in terms of the Boltzmann-type Ursell functions together with the rules taking quantum statistics into account (see Ref. [21] for the general formalism and Ref. [22] for the application to bosons). The many-body theory in terms of Ursell operators has been further studied in a series of papers [23] (see also Ref. [24]). Working in the occupation number representation in momentum space, also obtain formally exact integral equations for the average occupation of modes and, in particular, for the condensate density. The latter equation (a generalization of the Bogoliubov condensate equation) was shown to be exact in the thermodynamic limit by Ginibre [25], and it has also been established in the framework of infinitely extended states of the Bose systems [26]. Our formulation is different in the sense that it provides a closed equation for the density of the interacting gas at fixed chemical potential that is particularly well adapted to the exploration of the neighborhood of the mean-field limit. Moreover, it provides an interesting link with the theory of polymers.

Assume that the resummation of multiply connected diagrams leading to Eq. (49) can been performed. It is then not excluded that the resulting nonperturbative equation remains valid beyond the radius of convergence of the Mayer series. If so, the question of the Bose-Einstein condensation in an interacting gas could be examined on the basis of Eq. (49). At least one example supports this hope, i.e., such an extension to the transition region agrees with the rigorous results in the case of the mean-field limit. This fact motivates the study of the self-consistent relation (49) beyond the mean-field limit. We formulated this problem here using scaling (9) of the pair potential. In order to derive the form of function F_γ defining relation (49) for the scaled potential, one needs to know the small γ asymptotics of (i) the mean value of the

Boltzmann factor of a single polymer (or normalized partition function) and (ii) the sum of ring diagrams representing the contribution of the interacting polymers.

In the study of point (i) we could not use directly the known results (61) because of equal time condition imposed by quantum mechanics on interactions between the different elements of the polymers (20). The existence and localization of the Bose-Einstein condensation depends in a crucial way on the behavior of point (i) for extended polymers. This, however, remains an open problem. Our paper provides only the exact upper and lower bounds which, what is interesting, turn out to be qualitatively compatible with the classical result (61). There are some indications that in the $\gamma \to 0$ limit the quantum calculation approaches the classical one. However, this question is at present not yet understood. We have based the analysis of point (ii) on the remarkable fact that quantum Mayer ring diagrams can be summed in a closed analytical form. We thus arrived at an analytic expression (62) for the sum of the relevant ring diagrams. However, here again the small γ limit of Eq. (62) involves the normalized single polymer partition function whose asymptotics for the extended polymers remains to be derived. We hope that

the continuation of the study of quantum Mayer graphs along these lines and progress in the understanding of the polymer partition functions will eventually lead to the determination of functional relation (49) beyond the mean-field theory.

APPENDIX A: PROOF OF LEMMA 1

The average energy (57) is calculated from its definition (23) [introducing also the Fourier transform $\tilde{V}(\mathbf{k})$ of the potential]:

$$\langle U \rangle_q = J(q) - q \frac{V(0)}{2}, \tag{A1}$$

where

$$J(q) = \frac{1}{2} \int D_q(\mathbf{X}) \int_0^q ds_1 \int_0^q ds_2 \, \tilde{\delta}(s_1 - s_2) V(\lambda [\mathbf{X}(s_1) - \mathbf{X}(s_2)])$$

$$= \frac{1}{2} \int d\mathbf{k} \, \tilde{V}(\mathbf{k}) \int_0^q ds_1 \int_0^q ds_2 \, \tilde{\delta}(s_1 - s_2) \int D_q(\mathbf{X}) e^{i\lambda \mathbf{k} \cdot [\mathbf{X}(s_1) - \mathbf{X}(s_2)]}. \tag{A2}$$

From the basic rules for Fourier transforms of Gaussian measures, we have using the covariance (18)

$$\int D_q(\mathbf{X}) e^{i\lambda \mathbf{k} \cdot [\mathbf{X}(s_1) - \mathbf{X}(s_2)]} = \exp\left[-\frac{\lambda^2 k^2}{2} C_q(s_1 - s_2) \right],$$

$$k = |\mathbf{k}|, \tag{A3}$$

where

$$C_q(s) = |s| \left(1 - \frac{|s|}{q} \right). \tag{A4}$$

Since both $\tilde{\delta}(s)$ and $C_q(s)$ can be considered as periodic functions of period q, the double time integral in Eq. (A2) reduces to

$$\int_0^q ds_1 \int_0^q ds_2 \, \tilde{\delta}(s_1 - s_2) \exp\left[-\frac{\lambda^2 k^2}{2} C_q(s_1 - s_2) \right]$$

$$= q \sum_{n=0}^{q-1} \exp\left[-\frac{\lambda^2 k^2}{2} C_q(n) \right]$$

$$= q \sum_{n=1}^{q-1} \exp\left[-\frac{\lambda^2 k^2}{2} C_q(n) \right] + q. \tag{A5}$$

When this is introduced in Eqs. (A2) and (A1) [noting that $\int d\mathbf{k} \, \tilde{V}(\mathbf{k}) = V(0)$], one obtains finally

$$\langle U \rangle_q = q \int d\mathbf{k} \, \tilde{V}(\mathbf{k}) g(k, q) \tag{A6}$$

with

$$g(k, q) = \frac{1}{2} \sum_{n=1}^{q-1} \exp\left[-\frac{\lambda^2 k^2}{2} C_q(n) \right]$$

$$= \sum_{n=1}^{q/2} \exp\left[-\frac{\lambda^2 k^2}{2} C_q(n) \right]. \tag{A7}$$

The last equality results from the symmetry $C_q(n) = C_q(q - n)$. If $n \le q/2$, one has obviously $C_q(n) \ge n/2$ which leads to the bound

$$g(k, q) \le \sum_{n=1}^{q/2} \exp\left[-\frac{\lambda^2 k^2}{4} n \right] \le \frac{1}{\exp\left(\frac{\lambda^2 k^2}{4} \right) - 1} \tag{A8}$$

hence

$$\langle U \rangle_q \leq q \nu_+, \quad \nu_+ = \int d\mathbf{k} \frac{\tilde{V}(\mathbf{k})}{\exp\left(\frac{\lambda^2 k^2}{4}\right) - 1}, \quad (A9)$$

thus proving Lemma 1.

It is instructive to exhibit the behavior of ν_+ in the scaling limit of a Kac potential $\tilde{V}_\gamma(\mathbf{k}) = \tilde{V}(\mathbf{k}/\gamma)$. Setting $\mathbf{k} = \gamma \mathbf{p}$ in Eq. (A9) gives

$$\nu_+ = \gamma^3 \int d\mathbf{p} \frac{\tilde{V}(\mathbf{p})}{\exp\left(\frac{\gamma^2 \lambda^2 p^2}{4}\right) - 1} \sim \frac{4\gamma}{\lambda^2} \int d\mathbf{p} \frac{\tilde{V}(\mathbf{p})}{p^2}, \quad \gamma \to 0.$$

$$(A10)$$

APPENDIX B: PROOF OF LEMMA 2

We denote $\langle F \rangle_W = E(F|\mathbf{X}(0)=0)$ as the normalized Wiener expectation of the functional $F(\mathbf{X})$ of paths $\mathbf{X}(t)$ starting from the origin at time $t=0$, and $E(F|\mathbf{X}(t_1) = \mathbf{R}_1, \mathbf{X}(t_2) = \mathbf{R}_2)$ as the conditional Wiener expectation for paths starting in $\mathbf{R}_1$ at time t_1 and ending in $\mathbf{R}_2$ at time t_2. One has, in particular,

$$\langle F \rangle_W = \int d\mathbf{R}\, E(F|\mathbf{X}(0)=0, \mathbf{X}(t)=\mathbf{R}), \quad t > 0,$$

$$E(1|\mathbf{X}(t_1) = \mathbf{R}_1, \mathbf{X}(t_2) = \mathbf{R}_2) = \frac{\exp\left(-\frac{|\mathbf{R}_1 - \mathbf{R}_2|^2}{2(t_2 - t_1)}\right)}{(2\pi(t_2 - t_1))^{3/2}},$$

$$t_2 > t_1. \quad (B1)$$

The fact that Brownian motion is an homogeneous process implies the following symmetry relations under time and space translation, and space inversion:

$$E(F|\mathbf{X}(t_1) = \mathbf{R}_1, \mathbf{X}(t_2) = \mathbf{R}_2)$$

$$= E(F_\tau|\mathbf{X}(t_1 + \tau) = \mathbf{R}_1, \mathbf{X}(t_2 + \tau) = \mathbf{R}_2)$$

$$= E(F_\mathbf{R}|\mathbf{X}(t_1) = \mathbf{R}_1 + \mathbf{R}, \mathbf{X}(t_2) = \mathbf{R}_2 + \mathbf{R})$$

$$= E(F_-|\mathbf{X}(t_1) = -\mathbf{R}_1, \mathbf{X}(t_2) = -\mathbf{R}_2), \quad (B2)$$

with $F_\tau(\mathbf{X}(\cdots)) = F(\mathbf{X}(\cdots + \tau))$, $F_\mathbf{R}(\mathbf{X}(\cdots)) = F(\mathbf{X}(\cdots) + \mathbf{R})$ and $F_-(\mathbf{X}(\cdots)) = F(-\mathbf{X}(\cdots))$. Consider now the functional

$$F_{q_1,q_2}(\mathbf{X}) = \exp\left[-\frac{\beta}{2} \int_{q_1}^{q_2} ds \int_{q_1}^{q_2} dt\, \tilde{\delta}(s-t)(1 - \delta_{[s_1],[s_2]}^{Kr}) \right.$$

$$\left. \times V(\mathbf{X}(s) - \mathbf{X}(t)) \right], \quad (B3)$$

where index (q_1, q_2) means that the functional depends on the path only when t is in the interval $[q_1, q_2]$. In the present notation, the normalized Brownian bridge average of a functional F reads

$$\int D_q(\mathbf{X}) F(\mathbf{X}) = (2\pi q)^{3/2} E(F|\mathbf{X}(0)=0, \mathbf{X}(q)=0)$$

$$(B4)$$

and therefore the vertex function $\kappa(q)$ (54) for a q-particle loop $\mathbf{X}$ is

$$\kappa(q) = (2\pi q)^{3/2} E(F_{0,q}|\mathbf{X}(0)=0, \mathbf{X}(q)=0). \quad (B5)$$

In $F_{0,q}(\mathbf{X})$, we suppress the interaction between the sets of particles $1, \ldots, q_1$ and $q_1 + 1, \ldots, q$. Since V is positive this leads to the inequality

$$F_{0,q}(\mathbf{X}) \leq F_{0,q_1}(\mathbf{X}) F_{q_1,q}(\mathbf{X}), \quad (B6)$$

implying in Eq. (B5), in view of the Markov property of the Brownian motion,

$$\kappa(q) \leq (2\pi q)^{3/2} \int d\mathbf{R} E(F_{0,q_1}|\mathbf{X}(0)=0, \mathbf{X}(q_1)=\mathbf{R})$$

$$\times E(F_{q_1,q}|\mathbf{X}(q_1)=\mathbf{R}, \mathbf{X}(q)=0)$$

$$= (2\pi q)^{3/2} \int d\mathbf{R} E(F_{0,q_1}|\mathbf{X}(0)=0, \mathbf{X}(q_1)=\mathbf{R})$$

$$\times E(F_{0,q_2}|\mathbf{X}(0)=0, \mathbf{X}(q_2)=\mathbf{R}) \quad (B7)$$

with $q = q_1 + q_2$. The second line follows from the symmetry relations (B2) and the fact that the potential is invariant under space translations and inversion. Considering now the Wiener expectation of $F_{0,q}(\mathbf{X})$, we establish in the same way

$$\langle F_{0,q_1+q_2} \rangle_W \leq \langle F_{0,q_1} \rangle_W \langle F_{0,q_2} \rangle_W \quad (B8)$$

again as a consequence of the Markov property of the Brownian process and the invariance (B2). We exploit the inequalities (B7) and (B8) as follows. We first use Eq. (B7) to relax the constraint of the closed path, noting from Eq. (B1) that $E(F_{0,q_2}|\mathbf{X}(0)=0, \mathbf{X}(q_2)=\mathbf{R}) \leq (2\pi q_2)^{-3/2}$ and from Eq. (B7)

$$\kappa(q) \leq \left(1 + \frac{q_1}{q_2}\right)^{3/2} \langle F_{0,q_1} \rangle_W, \quad q = q_1 + q_2, \quad q_2 \geq 1.$$

$$(B9)$$

Then we set $q_1 = nq_0, q_2 = n, q = n(q_0 + 1)$ for some fixed integer $q_0 \geq 1$. From Eq. (B9) and the iteration of Eq. (B8)

$$\kappa(n(q_0 + 1)) \leq (1 + q_0)^{3/2} (\langle F_{0,q_0} \rangle_W)^n. \quad (B10)$$

If one sets

$$\nu_- = -\frac{1}{\beta(q_0 + 1)} \ln \langle F_{0,q_0} \rangle_W > 0, \quad r = q_0 + 1, \quad (B11)$$

one obtains the result of Lemma 2.

APPENDIX C: THE BOSE GAS IN AN EXTERNAL POTENTIAL

Motivated by the experimental situation of bosonic atoms in traps, we briefly show in this appendix how the formalism works in presence of an external field (more details can be found in Ref. [27]). If a one-body external potential $V^{\text{ext}}(\mathbf{r})$ is introduced, the formula for the grand-canonical partition function Ξ is still given by Eq. (14) with the only change that the loop activity $z(\mathcal{L})$ is replaced by

$$\tilde{z}(\mathcal{L}) = z(\mathcal{L}) e^{-\beta V^{\text{ext}}(\mathcal{L})},$$

$$V^{\text{ext}}(\mathcal{L}) = \int_0^q ds\, V^{\text{ext}}(\mathbf{R} + \lambda \mathbf{X}(s)). \qquad (C1)$$

Since the density is nonuniform, it is appropriate to consider here the average total particle number $\langle N \rangle$:

$$\langle N \rangle = \beta^{-1} \frac{\partial}{\partial \mu} \ln \Xi = \sum_{n=1}^{\infty} \frac{1}{n!} \int d\mathcal{L}_1 \cdots d\mathcal{L}_n \left(\sum_{\ell=1}^n q_\ell \right)$$

$$\times \prod_{k=1}^n \tilde{z}(\mathcal{L}_k) u(\mathcal{L}_1, \ldots, \mathcal{L}_n)$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int d\mathcal{L}_1 \cdots d\mathcal{L}_n q_1$$

$$\times \prod_{k=1}^n \tilde{z}(\mathcal{L}_k) u(\mathcal{L}_1, \ldots, \mathcal{L}_n), \qquad (C2)$$

where we have introduced the Mayer expansion of $\ln \Xi$ and $u(\mathcal{L}_1, \ldots, \mathcal{L}_n)$ is the Ursell function (31).

The external potential will be chosen positive and confining with the sufficiently fast growth at infinity so that $\int d\mathbf{r} \exp(-\beta V^{\text{ext}}(\mathbf{r})) < \infty$. In order to allow for a well-defined infinite particle number limit, it is necessary to scale the external potential as

$$V^{\text{ext}}_\omega(\mathbf{r}) = V^{\text{ext}}(\omega \mathbf{r}), \qquad (C3)$$

with $V^{\text{ext}}(\mathbf{r})$ a fixed positive confining potential. As the scaling parameter ω tends to 0, $V^{\text{ext}}_\omega(\mathbf{r})$ becomes less confining and the average particle number $\langle N \rangle(\beta, \mu, \omega)$ diverges. The proper quantity that remains finite in the limit is the product $\omega^3 \langle N \rangle(\beta, \mu, \omega)$. The experimental situation for atoms in traps appears to be well described in this asymptotic regime [1]. We have indeed the following proposition.

Proposition 5. In the range of convergence of the Mayer series for the uniform Bose gas, $\lim_{\omega \to 0} \omega^3 \langle N \rangle(\beta, \mu, \omega) \equiv \mathcal{N}(\beta, \mu)$ exists and

$$\mathcal{N}(\beta, \mu) = \int d\mathbf{r}\, \rho(\beta, \mu - V^{\text{ext}}(\mathbf{r})), \qquad (C4)$$

with $\rho(\beta, \mu)$ the density of the uniform Bose gas.

Proof. In the scaling limit defined by Eq. (C3), the gas is locally uniform at point $\mathbf{r}$ with a space dependent chemical potential $\mu(\mathbf{r}) \equiv \mu - V^{\text{ext}}(\mathbf{r})$.

We write $V^{\text{ext}}_\omega(\mathcal{L}) = V^{\text{ext}}(\omega \mathbf{R}, q, \omega \mathbf{X})$ and $u(\mathcal{L}_1, \ldots, \mathcal{L}_n) = u(\mathbf{R}_1, q_1, \mathbf{X}_1, \ldots, \mathbf{R}_n, q_n, \mathbf{X}_n)$. The spatial part of the integral in the nth order term of the series (C2) reads (with the additional factor ω^3)

$$\omega^3 \int d\mathbf{R}_1\, d\mathbf{R}_2 \cdots d\mathbf{R}_n \prod_{k=1}^n e^{-\beta V^{\text{ext}}(\omega \mathbf{R}_k, q_k, \omega \mathbf{X}_k)} u(\mathbf{R}_1, q_1, \mathbf{X}_1, \mathbf{R}_2, q_2, \mathbf{X}_2, \ldots, \mathbf{R}_n, q_n, \mathbf{X}_n)$$

$$= \int d\mathbf{r}\, e^{-\beta V^{\text{ext}}(\mathbf{r}, q_1, \omega \mathbf{X}_1)} \int d\mathbf{R}_2 \cdots d\mathbf{R}_n \prod_{k=2}^n e^{-\beta V^{\text{ext}}(\mathbf{r} + \omega \mathbf{R}_k, q_k, \omega \mathbf{X}_k)} u(0, q_1, \mathbf{X}_1, \mathbf{R}_2, q_2, \mathbf{X}_2, \ldots, \mathbf{R}_n, q_n, \mathbf{X}_n), \qquad (C5)$$

where one has made the successive change of variables $\mathbf{R}_k \to \mathbf{R}_k + \mathbf{R}_1$, $k = 2, \ldots, n$, then $\omega \mathbf{R}_1 = \mathbf{r}$, and used that the Ursell function is translation invariant with respect to its spatial variables. Since $\lim_{\omega \to 0} V^{\text{ext}}(\mathbf{r} + \omega \mathbf{R}_k, q_k, \omega \mathbf{X}_k) = q_k V^{\text{ext}}(\mathbf{r})$ and the Ursell function is jointly integrable on $\mathbf{R}_2, \ldots, \mathbf{R}_k$, expression (C5) tends by dominated convergence to

$$\int d\mathbf{r} \left[\prod_{k=1}^n e^{-q_k V^{\text{ext}}(\mathbf{r})} \right] \left[\int d\mathbf{R}_2 \cdots d\mathbf{R}_n\, u(0, \mathbf{X}_1, q_1, \mathbf{R}_2, \mathbf{X}_2, q_2, \ldots, \mathbf{R}_n, \mathbf{X}_n, q_n) \right]. \qquad (C6)$$

The first bracket, when combined with the loop activity $z(\mathcal{L})$ (22), simply yields the shifted local chemical potential $\mu(\mathbf{r}) = \mu - V^{\text{ext}}(\mathbf{r})$, whereas the second bracket is the spatial part of the integral of the Ursell function occurring in the loop density series (29) of the uniform gas evaluated at point $\mathbf{R}_1 = 0$. When this is introduced in the series (C2) for

$\omega^3 \langle N \rangle(\beta, \mu, \omega)$ and the integrations on the internal degrees of freedom of the loops are taken into account [using also Eq. (32)], one obtains the result (C4) in the limit $\omega \to 0$.

To show convergence, one introduces the expression (C5) in the series (C2) and majorize factors $e^{-\beta V^{\text{ext}}(\mathbf{r} + \omega \mathbf{R}_k, q_k, \omega \mathbf{X}_k)}$ by 1, $k = 2, \ldots, n$:

$$\omega^3 \langle N \rangle (\beta, \mu, \omega) \leq \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \sum_{q_1=1}^{\infty} q_1 \int D_{q_1}(\mathbf{X}_1)$$

$$\times \int d\mathbf{r}\, e^{-\beta V^{\text{ext}}(\mathbf{r},q_1,\omega \mathbf{X}_1)} \prod_{k=1}^{n} z(\mathcal{L}_k)$$

$$\times \int d\mathcal{L}_2 \cdots d\mathcal{L}_n u(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_n)$$

$$= \sum_{q_1}^{\infty} q_1 \int D_{q_1}(\mathbf{X}_1) \int d\mathbf{r}\, e^{-\beta V^{\text{ext}}(\mathbf{r},q_1,\omega \mathbf{X}_1)}$$

$$\times \rho_{\text{loop}}(\mathcal{L}_1), \tag{C7}$$

where $\rho_{\text{loop}}(\mathcal{L}_1)$ is the Mayer series (29) for the uniform system evaluted at $\mathcal{L}_1 = (0, q_1, \mathbf{X}_1)$. Using again the positivity of $V^{\text{ext}}(\mathbf{r})$ and Jensen inequality, we have

$$\int D_{q_1}(\mathbf{X}_1) \int d\mathbf{r}\, e^{-\beta V^{\text{ext}}(\mathbf{r},q_1,\omega \mathbf{X}_1)}$$

$$\leq \int D_{q_1}(\mathbf{X}_1) \int d\mathbf{r} \exp\left[-\beta \int_0^1 ds\, V_{\text{ext}}(\mathbf{r} + \omega \lambda \mathbf{X}(s)) \right]$$

$$\leq \int D_{q_1}(\mathbf{X}_1) \int_0^1 ds \int d\mathbf{r} \exp[-\beta V_{\text{ext}}(\mathbf{r} + \omega \lambda \mathbf{X}(s))]$$

$$= \int d\mathbf{r}\, e^{-\beta V^{\text{ext}}(\mathbf{r})}. \tag{C8}$$

The convergence of Mayer series (29) is established in Proposition 2 with bounds that are independent of the shape $\mathbf{X}_1$ of the loop $\mathcal{L}_1$. Combining this previous analysis with Eq. (C8), one gets eventually

$$\omega^3 \langle N \rangle (\beta, \mu, \omega) \leq \bar{\rho}(\mu) \int d\mathbf{r}\, e^{-\beta V^{\text{ext}}(\mathbf{r})} \tag{C9}$$

with $\bar{\rho}(\mu)$ defined in Proposition 2. Since estimates are uniform with respect to ω, the existence of the limit (C4) follows again by the dominated convergence.

Moreover, if the two-body potential is scaled according to Eq. (9), one obtains immediately from Proposition 3 the mean-field limit for the trapped Bose gas

$$\mathcal{N}_{\text{mf}}(\beta, \mu) \equiv \lim_{\gamma \to 0} \mathcal{N}(\beta, \mu, \gamma) = \int d\mathbf{r}\, \rho_{\text{mf}}(\beta, \mu - V^{\text{ext}}(\mathbf{r})). \tag{C10}$$

This result is established by other methods in Ref. [9] in a slightly different situation. The external potential considered in Ref. [9] has support in a box of volume L^3 and is scaled according to the size of the box [namely, $\omega = 1/L$ in Eq. (C3)].

The thermodynamics of the mean-field trapped gas can be studied in detail from Eq. (C10). If $\mu \leq \mu_c = a\rho_{0,c}$ (the critical chemical potential of the homogeneous mean-field gas), then $\mu(\mathbf{r}) = \mu - V^{\text{ext}}(\mathbf{r}) \leq \mu_c$ for all $\mathbf{r}$, so condensation does not occur anywhere. If $\mu > \mu_c$, there is a local condensate of density $[\mu(\mathbf{r})/a - \rho_{0,c}]$ at all points $\mathbf{r}$ such that $\mu(\mathbf{r}) > \mu_c$, namely in the region of space $\Delta = \{\mathbf{r} | V^{\text{ext}}(\mathbf{r}) < \mu - \mu_c\}$. Then the total amount of condensate is

$$\mathcal{N}_{\text{cond}}(\beta, \mu) = \int_\Delta d\mathbf{r}[\mu(\mathbf{r})/a - \rho_{0,c}]. \tag{C11}$$

Obviously $\mathcal{N}_{\text{cond}}(\beta, \mu)$ vanishes at $\mu = \mu_c$ so the critical temperature $T_c(\mathcal{N})$ as function of the total particle number is defined as the solution of Eq. (C10) when $\mu = \mu_c$ and $\mathcal{N}$ is fixed, namely

$$\mathcal{N}_{\text{mf}}(\beta_c, \mu(\beta_c)) = \mathcal{N}, \quad \mu(\beta) = a\rho_{0,c}(\beta). \tag{C12}$$

As an example, for an harmonic potential $V^{\text{ext}}(\mathbf{r}) = b|\mathbf{r}|^2/2$, one finds a critical behavior of the condensate fraction $\mathcal{N}_{\text{cond}}(\beta, \mathcal{N})/\mathcal{N}$ of the form [keeping now the particle number $\mathcal{N} = \mathcal{N}_{\text{mf}}(\beta, \mu)$ fixed] [27]

$$\frac{\mathcal{N}_{\text{cond}}(\beta, \mathcal{N})}{\mathcal{N}} \sim \left(1 - \frac{T}{T_c}\right)^{5/2}, \quad T \to T_c. \tag{C13}$$

This is to be contrasted with the behaviors of the same fraction for the homogeneous free or mean-field gas [$\sim 1 - (T/T_{0,c})^{3/2}$], and for the free gas in an harmonic potential [$\sim 1 - (T/T_{\text{trap},c})^3$] when T approaches the corresponding critical temperatures.

Clearly, when the corrections to the mean field density of the homogeneous gas are known, they can be implemented in Eq. (C4) and thus will also give interesting informations on the imperfect trapped gases.

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